

Chapter 4

Deformation of cylinder knots

4.1 Introduction

In Chapters 1 - 3 we studied Lissajous knots and their diagrams, symmetric unions and cylinder knots.¹ We learned that a cylinder knot with coprime parameters n and m is a symmetric union.

In this chapter we define a family $R(n_1, n_2, n_3)$ of Lissajous knots which are symmetric unions. Then we succeed in establishing a relationship between Lissajous knots and cylinder knots: if certain conditions are satisfied the partial knot of $R(n_1, n_2, n_3)$ is the same as the partial knot of $Z(2n_1, n_2, n_3)$. Although these knots are different in general, by Theorem 2.6 in [Ch2] their determinants are equal.

The second topic of this chapter is the possibility to exchange the parameters n and m of cylinder knots. The examples $Z(3, 11, 13) = Z(3, 13, 11)$ and $Z(3, 11, 16) = Z(3, 16, 11)$, but $Z(4, 11, 13) \neq Z(4, 13, 11)$ lead to the

Question 4.1.1

Under which circumstances is $Z(s, n, m)$ equal to $Z(s, m, n)$?

To answer this question and to analyze the first problem we consider billiard knots in a solid torus. This solid torus we choose as a cube with identified front and back face (a “flat solid torus”). The billiard knots in it have the parameters s, n, m as known from cylinder knots. But here n and m are exchangeable.

We develop a procedure to deform billiard curves in a cylinder to billiard curves in a flat solid torus. During the deformation singularities can occur and the knot type can change. We can express the result of our investigation in a simplified way like this: if both billiard curves have the property that their knot type is the same at the beginning and the end of the deformation (we say the curve is “weakly stable”) then $Z(s, n, m) = Z(s, m, n)$.

¹These chapters correspond to articles [Ch1a], [Ch1b], [Ch2] and [Ch3].

4.2 Lissajous knots which are symmetric unions

We start with the definition of the knots $R(n_1, n_2, n_3)$ which are a subfamily of Lissajous knots (or billiard knots in a cube). As in Chapter 3 let g be the sawtooth function $g(t) = 2|t - \lfloor t \rfloor - \frac{1}{2}|$.

Definition 4.2.1

Let n_2 be odd and $R(n_1, n_2, n_3)$ be the billiard knot in a cube defined by the map $t \mapsto (g(n_1 t + \frac{1}{4}), g(n_2 t), g(n_3 t + \gamma))$, $t \in [0, 1]$, with $\gamma > 0$ sufficiently small.

The meaning of the condition “sufficiently small” will become clear in the following lemma: the phase γ is moved away from zero so that the singularities in the knot are resolved and no new ones are added.

Lemma 4.2.2

The knots $R(n_1, n_2, n_3)$ are symmetric unions.

PROOF: Because of $g(\frac{1}{4}) = \frac{1}{2}$ and $g(0) = 1$, the projection on the x - y -plane is symmetric to the axis $\{(x, y) | x = \frac{1}{2}\}$. If $\gamma = 0$ there is a maximum at $(\frac{1}{2}, 1)$ and the crossings on the symmetry axis are singular. Lemma 1.2 in [JP] shows that there are no other singularities. A small deformation from $\gamma = 0$ to $\gamma > 0$ yields a symmetric union. \square

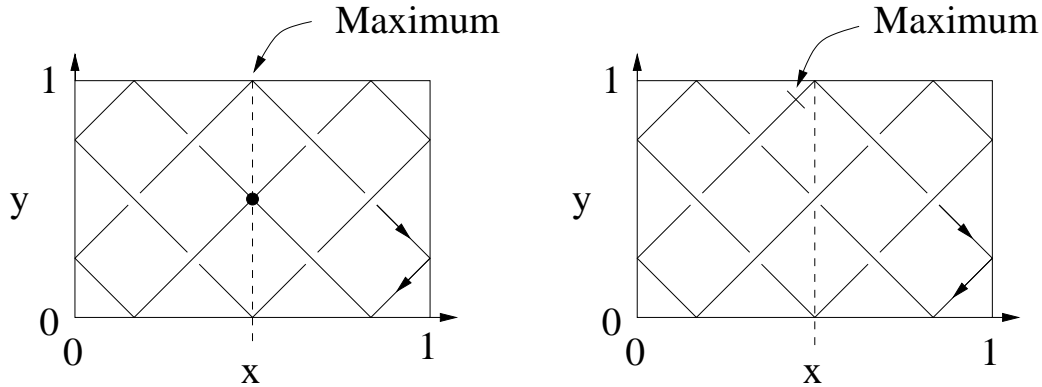


Figure 4.1: The construction of billiard knots in a cube which are symmetric unions.

For some examples of knots from the families $Z(2s, n, m)$ and $R(s, n, m)$ we computed the determinants. For $s = 2$ and $n = 11$ we get the following results:

m	$\det Z(4, 11, m)$	$\det R(2, 11, m)$
35	1	99^2
37	1	571^2
39	113^2	113^2
41	1	329^2
43	187^2	187^2

As we mentioned in the introduction² these numbers made us ask the

Question 4.2.3

Is there a relationship between the families Z and R ?

The two question 4.1.1 and 4.2.3 started the analysis in this chapter and they were the motivation to prove the determinant formula in Theorem 2.6 in [Ch2]. Our answers are contained in the Corollaries 4.4.3 and 4.4.5.

4.3 Billiard knots in a flat solid torus

Definition 4.3.1

We call $VT = \mathbf{I}^3 / (0, y, z) \sim (1, y, z)$, the cube with identified front and back face, a *flat solid torus* and the periodic billiard curves in it *billiard knots in a flat solid torus*.

A notation for billiard knots in a flat solid torus is

$$\begin{aligned}
 T(s, n, m, \phi) : [0, 1] &\rightarrow VT \\
 t &\mapsto (s \cdot t \bmod 1, g(nt), g(mt + \phi)).
 \end{aligned}$$

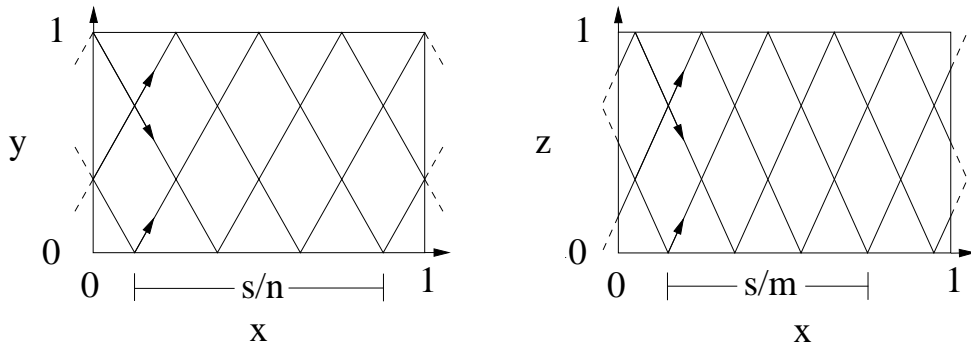


Figure 4.2: The parameters of billiard knot in a flat solid torus. The figures show the projections on the x - y -plane and the x - z -plane, respectively.

²Not part of this chapter.

The integers s, n, m are the number of strings, the number of reflections to the respective faces and $\phi \in [0, 1]$ is a phase. We need $\gcd(s, n) = \gcd(s, m) = 1$ (the parameters n and m are symmetric to each other) but the condition $n \geq 2s + 1$ of the cylinder theory is not necessary here.

Lemma 4.3.2

- (i) *There is always a phase ϕ such that $T(s, n, m, \phi)$ is without self-intersections.*
- (ii) *The knots $T(s, n, m, \phi)$ are independent of the phase (up to taking mirror image).*

PROOF: (i): A computation of the crossing parameters shows that if all $\phi \in [0, 1]$ lead to singular knots we have $\frac{mk}{s} \in \mathbf{Z}$ with a $k \in \{1, \dots, s-1\}$. Hence if $\gcd(s, m) = 1$ and $s \geq 2$ this is impossible. If $s = 1$ there are no crossings at all in the projection. For the proof of (ii) we use (i) and the same symmetry argument as in Lemma 2.3 in [Ch3]. \square

By the first part of the lemma we can neglect the phase and just write $T(s, n, m)$. The knots in this chapter are unique only up to mirror image. Therefore we write $K_1 = K_2$, if K_1 is equivalent to K_2 , $-K_2$, K_2^* or $-K_2^*$.

The proofs of the next theorems are analogous to the respective proofs in the last chapter because all of them are based on the dihedral symmetry of the projection curve. We first treat the aperiodic case:

Theorem 4.3.3

For pairwise coprime parameters s, n, m the knot $T(s, n, m)$ is a ribbon knot and we have $T(s, n, m) = T(s, m, n)$. If n or m are even then $T(s, n, m)$ is strongly positive amphicheiral. \square

We return to the general case with $\gcd(n, m) = d \geq 1$.

Theorem 4.3.4

If $\gcd(n, m) = d$ then $T(s, n, m)$ has cyclic period d with linking number s . If n is odd and m is even or vice versa, $T(s, n, m)$ is strongly positive amphicheiral. The parameters n and m are exchangeable: we have $T(s, n, m) = T(s, m, n)$. The factor knot corresponding to the cyclic symmetry is $T(s, \frac{n}{d}, \frac{m}{d})$. It is a symmetric union. \square

Further analysis of the situation shows the

Theorem 4.3.5

- If n and m are odd then*
 - if s is even $T(s, n, m)$ has cyclic period 2,*
 - if s is odd $T(s, n, m)$ has free period 2.*

PROOF: We replace in the parametrization t by $t + \frac{1}{2}$. The result is $(s(t + \frac{1}{2}) \bmod 1, \frac{1}{2} - g(nt), \frac{1}{2} - g(mt + \alpha))$ because the function g has the property $g(t + \frac{1}{2}) = \frac{1}{2} - g(t)$.

If s is even this shows that the knot is mapped to itself by a rotation of π around the circle $\{(x, y, z) | y = z = \frac{1}{2}\}$. A non-singular knot is disjoint from this circle, hence it has cyclic period 2.

If s is odd the knot is invariant under the fixpoint-free map $(x, y, z) \mapsto (x + \frac{1}{2}, \frac{1}{2} - y, \frac{1}{2} - z)$, hence it is freely periodic with period 2. \square

Remark 4.3.6

If s is even, n and m must be odd, so all knots $T(s, n, m)$ with even s have cyclic period 2.

Remark 4.3.7

We consider the knots which are parametrized in cylinder coordinates by

$$(\varphi(t), r(t), z(t)) = (st, 3 + \cos(nt), \cos(mt + \phi)), \quad t \in [0, 2\pi]. \quad (4.1)$$

They are identical to the knots $T(s, n, m, \phi)$:

We deform the function cosine as we did in the case of Lissajous knots to a piecewise linear function. Hence we can write

$$(\varphi(t), r(t), z(t)) = (2\pi \cdot st, 1 + g(nt), g(mt + \phi)), \quad t \in [0, 1]. \quad (4.2)$$

We map the flat solid torus $KR = \{(\varphi, r, z) | \varphi \in [0, 2\pi], 1 \leq r \leq 2, 0 \leq z \leq 1\}$ by the homeomorphism $(\varphi, r, z) \mapsto (\varphi/2\pi, r - 1, z)$ to VT . The image of a knot as (4.2) is the knot in the flat solid torus $T(s, n, m, \phi)$. An example for (4.1) with $s = 3$, $n = 7$, $m = 5$ can be seen in Figure 4.3.

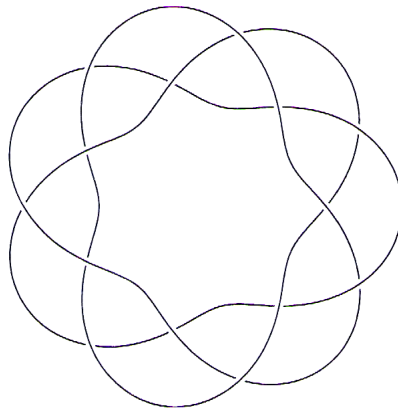


Figure 4.3: The knot $T(3, 7, 5)$ parametrized in cylinder coordinates.

4.4 Stable cylinder knots

Given an integer a coprime to s the knot $Z(s, an, am)$ has cyclic period a . The factor knot $Z(s, an, am)^{(a)}$ is similar to $Z(s, n, m)$: it consists of the same numbers of strings, reflections to the boundary $\mathbf{S}^1 \times \mathbf{I}$ and maxima. We expect that these knots “converge” in some way for $a \rightarrow \infty$ (see Figure 4.4). Thus if the knots $Z(s, an, am)^{(a)}$ are independent of a for large a we write suggestively $Z^{st}(s, n, m) = \lim_{a \rightarrow \infty} Z(s, an, am)^{(a)}$ for the limit knot.

Instead of using the factor knot of the periodic knot with period a , we identify the radial faces of a cylinder’s $\frac{2\pi}{a}$ -slice to define $Z(s, an, am)^{(a)}$. This description allows us to construct the deformed knots for a which are not coprime to s ; we also use the notation $Z(s, an, am)^{(a)}$. The condition $n \geq 2s + 1$ for cylinder knots is unnecessary for the knots Z^{st} because we can choose a so that $an \geq 2s + 1$.

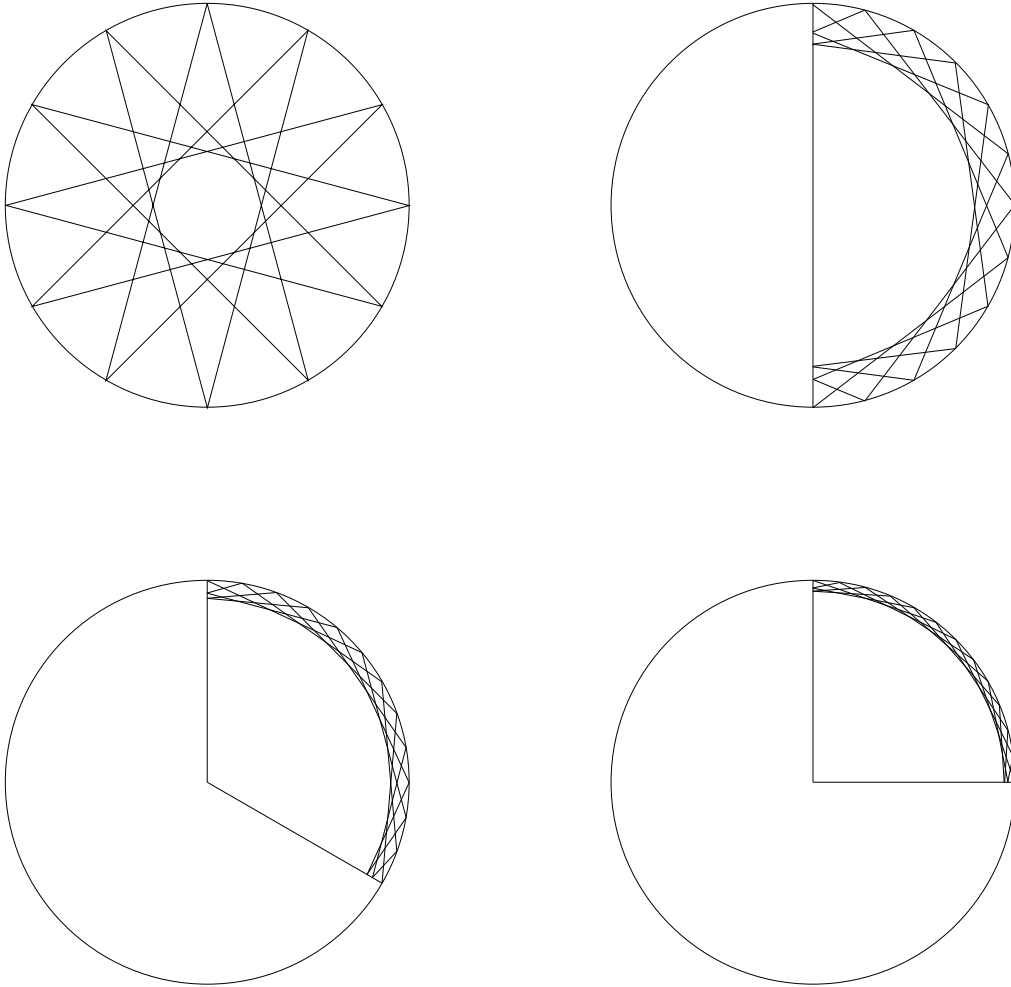


Figure 4.4: The projected billiard curves for $s = 5$, $n = 12$, $a = 1, 2, 3, 4$.

Theorem 4.4.1

Let s, n and m be integers with $\gcd(s, n) = 1$. Then the knot $Z^{st}(s, n, m) := \lim_{a \rightarrow \infty} Z(s, an, am)^{(a)}$ is well-defined: there is an integer \tilde{a} so that for all a with $a \geq \tilde{a}$ we have $Z(s, \tilde{a}n, \tilde{a}m)^{(\tilde{a})} = Z(s, an, am)^{(a)}$.

We call the knots $Z^{st}(s, n, m)$ *stable cylinder knots*. We can describe stable cylinder knots in terms of the above defined billiard knots in a flat solid torus if the parameters allow it, that is if s and m are coprime.

Theorem 4.4.2

If the billiard knot in a flat solid torus T is defined for the parameters s, n, m then $Z^{st}(s, n, m) = T(s, n, m)$. The billiard knots in a flat solid torus are a subfamily of the knots Z^{st} .

Answer to Question 4.1.1

Now we give an answer to Question 4.1.1. A billiard curve in the cylinder with $Z(s, n, m) = Z^{st}(s, n, m)$ is called *weakly stable*. In this case we also say that $Z(s, n, m)$ is weakly stable, though this property is not a knot invariant but a property of the billiard curve. Our theorems yield the following Corollary 4.4.3, for which we assume $\gcd(s, n) = \gcd(s, m) = 1$ and $n, m \geq 2s + 1$.

Corollary 4.4.3

If the cylinder knots $Z(s, n, m)$ and $Z(s, m, n)$ are weakly stable then we have $Z(s, n, m) = Z(s, m, n)$. \square

Answer to Question 4.2.3

Our second Question 4.2.3 is answered by Theorem 4.4.4 and Corollary 4.4.5.

Theorem 4.4.4

If $2s, n, m$ are pairwise coprime, then the knots $T(2s, n, m)$ and $R(s, n, m)$ are symmetric unions with identical partial knots. Hence the determinant of $T(2s, n, m)$ equals the determinant of $\det R(s, n, m)$.

From this we deduce immediately the

Corollary 4.4.5

Let $2s, n, m$ be pairwise coprime and $n \geq 4s + 1$. If $Z(2s, n, m)$ is weakly stable then $\det Z(2s, n, m) = \det R(s, n, m)$. \square

Remark 4.4.6

The relationships between the knot families in this chapter are summarized in the following diagram.

cylinder knots,
 $Z(s, n, m)$

Lissajous knots

\downarrow

\cup

stable cylinder
 knots ,
 $Z^{st}(s, n, m)$

\supset

billiard knots
 in a flat solid
 torus, $T(s, n, m)$

\longleftrightarrow

Lissajous knots,
 which are symmetric
 unions, $R(n_1, n_2, n_3)$

4.4.1 The deformation process

In the construction of the stable cylinder knots we used slices of the cylinder with angles $\beta = 2\pi/a$. However, to be more flexible we now vary the angle β continuously from 2π to zero.

We set the total length of the projected curve κ onto the bottom of the slice to 1 and parametrize κ according to its arclength. The base point $\kappa(0) = \kappa(1)$ is chosen to be a vertex on the boundary of \mathbf{D}^2 . The crossings of κ are numbered from 1 to $(s-1)n$ and the parameters of the i -th crossing are $t_i(\beta)$ and $t'_i(\beta)$. For the length of P_0P_b in the figure on page 356 in [Ch3] depending on β we write $\frac{x_b(\beta)}{2n}$. Then

$$t_i(\beta) = \frac{k_i}{2n} + \frac{x_{l_i}(\beta)}{2n}, \quad t'_i(\beta) = \frac{k'_i}{2n} - \frac{x_{l_i}(\beta)}{2n},$$

with $k_i, k'_i, l_i \in \mathbb{N}$. It is remarkable that the sum $t_i(\beta) + t'_i(\beta)$ is independently of β a constant multiple of $1/2n$.

Let $\Delta_i^\phi(\beta)$ be the height difference of the billiard curve at the crossing i as a function of β and phase $\phi \in [0, 1]$. Then for $\Delta_i^\phi(\beta)$ we obtain the expression

$$\Delta_i^\phi(\beta) := g(mt_i(\beta) + \phi) - g(mt'_i(\beta) + \phi).$$

The sign of $\Delta_i^\phi(\beta)$

Since we are interested only in the sign of $\Delta_i^\phi(\beta)$ we compare the function $g(t)$ with $h(t) = (\cos(2\pi t) + 1)/2$ in the following way in order to use a theorem from trigonometry:

$$\begin{aligned} g(t_1) > g(t_2) &\Leftrightarrow h(t_1) > h(t_2), \\ g(t_1) = g(t_2) &\Leftrightarrow h(t_1) = h(t_2), \\ g(t_1) < g(t_2) &\Leftrightarrow h(t_1) < h(t_2). \end{aligned}$$

The formula $\cos(t_1) - \cos(t_2) = -2 \sin((t_1 + t_2)/2) \sin((t_1 - t_2)/2)$ yields

$$\text{sign } \Delta_i^\phi(\beta) = -\text{sign} \left(\sin(\pi[m[t_i(\beta) + t'_i(\beta)] + 2\phi]) \cdot \sin(\pi m[t_i(\beta) - t'_i(\beta)]) \right).$$

This is useful because the first factor is independent of β and the second is independent of ϕ .

If ϕ is chosen so that $\Delta_i^\phi(2\pi) \neq 0$, we can form the quotient

$$\delta_i(\beta) = \text{sign} \frac{\Delta_i^\phi(\beta)}{\Delta_i^\phi(2\pi)} = \text{sign} \frac{\sin(\pi m[t_i(\beta) - t'_i(\beta)])}{\sin(\pi m[t_i(2\pi) - t'_i(2\pi)])}. \quad (4.3)$$

If this sign is $+1$ the sign of the crossing i is the same for the angles β and 2π , if it is -1 a crossing change has happened and if it is zero the crossing is singular for the angle β . We stress the fact that the phase ϕ is not contained in the expression.

Stability of cylinder knots

After these technical preparations we can define several notions of stability. As in the definition of weakly stable cylinder knots before Corollary 4.4.3, the following definition is about properties of billiard curves in a cylinder, not about knot invariants.

Definition 4.4.7

The cylinder knot $Z(s, n, m)$ is called

$$\begin{aligned} \text{weakly stable} & :\Leftrightarrow Z(s, n, m) = Z^{st}(s, n, m), \\ \text{positively (negatively) stable} & :\Leftrightarrow \exists \varepsilon > 0 \text{ so that for all } i \text{ and for all } \beta \in]0, \varepsilon] : \\ & \frac{\Delta_i^\phi(\beta)}{\Delta_i^\phi(2\pi)} > 0 \text{ } (< 0), \\ \text{strongly positive stable} & :\Leftrightarrow \text{for all } i \text{ and for all } \beta \in]0, 2\pi] : \\ & \frac{\Delta_i^\phi(\beta)}{\Delta_i^\phi(2\pi)} > 0. \end{aligned}$$

If the cylinder knot is positively or negatively stable we call it also just *stable*. Stable knots are weakly stable and, of course, strongly positive stable knots are positively stable.

Examples 4.4.8

(i) $Z(p, q, q)$ is the torus knot $t(p, q)$ (see [JP], p. 153/154). It is strongly positive stable because the distribution of q maxima on the q chords of the projection can be chosen independently of β . For instance if we place the maximum and minimum at the ratios $1/4$ and $3/4$ of the chord's length we get the torus knot $t(p, q)$ for all $\beta \in]0, 2\pi]$.

(ii) If the parameters $4, n, m$ are pairwise coprime and the knot $Z(4, n, m)$ is negatively enlaced³ then it is not stable. By Theorem 4.3.5 we know that the curves $Z^{st}(4, n, m) = T(4, n, m)$ are positively enlaced.

³Using the notation of Theorem 4.2 in [Ch3] we call the cases $v_1 = -v_3$ and $v_1 = v_3$ *negatively enlaced* and *positively enlaced*, respectively.

4.4.2 Proofs of Theorems 4.4.1, 4.4.2 and 4.4.4

PROOF OF THEOREM 4.4.1: As in Proposition 2.5 in [Ch3] we compute the parameters $x_l(\beta)$ as

$$x_l(\beta) = \frac{\tan(l \frac{\beta}{2n})}{\tan(s \frac{\beta}{2n})}.$$

It is not difficult to show that $\frac{d}{d\beta} x_l(\beta) < 0$ for $\beta \in]0, 2\pi]$, and with de L'Hospital's rule we get $\lim_{\beta \rightarrow 0} x_l(\beta) = \frac{l}{s}$.

The difference $t_i(\beta) - t'_i(\beta)$ is equal to $\frac{k_i - k'_i}{2n} + \frac{2x_{l_i}(\beta)}{2n}$ and hence strictly monotonously decreasing. Hence for all i we find an $\varepsilon_i > 0$ so that $\text{sign}(\sin(\pi m[t_i(\beta) - t'_i(\beta)]))$ is independent of β for all $\beta \in]0, \varepsilon_i]$. For all a with $\frac{1}{a} < \min_i \varepsilon_i$ the knots $Z(s, an, am)^{(a)}$ are the same (up to mirror image, as usual), and we denote them as

$$\lim_{a \rightarrow \infty} Z(s, an, am)^{(a)} =: Z^{st}(s, n, m)$$

as we did in the formulation of the theorem. \square

PROOF OF THEOREM 4.4.2: The knots $T(s, n, m)$ are exactly the limit case $\beta = 0$. To show this we look at Figure 4.5. There we chose the same notation as in the figure on page 356 in [Ch3].

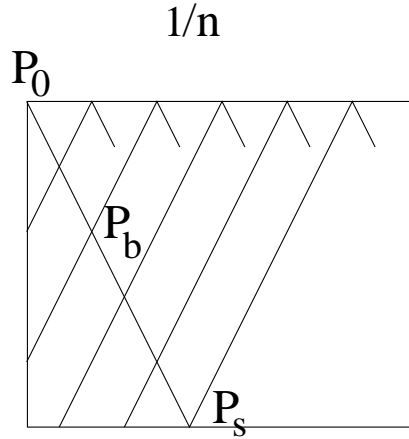


Figure 4.5: The computation of $|P_0 P_b| / |P_0 P_s|$

We have $|P_0 P_b| / |P_0 P_s| = \frac{b}{s}$. Therefore from $\lim_{\beta \rightarrow 0} x_b(\beta) = \frac{b}{s}$ we deduce that the knot $T(s, n, m)$ is equal to $Z^{st}(s, n, m)$. \square

PROOF OF THEOREM 4.4.4: We consider the knots $T(2s, n, m)$ and $R(s, n, m)$ with a maximum pushed to the axis of symmetry.

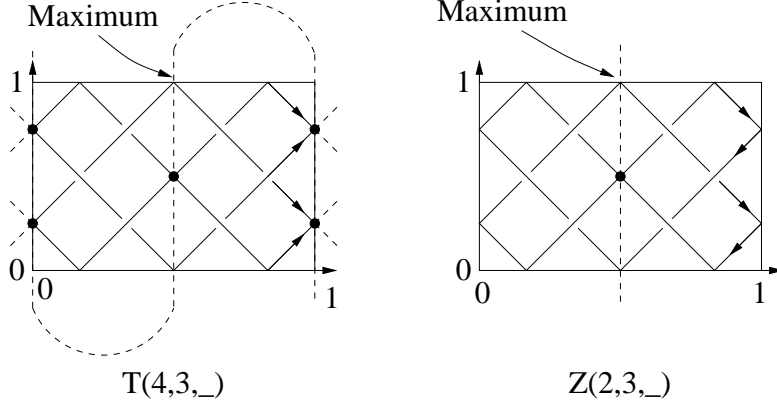


Figure 4.6: Diagrams of the knots $T(2s, n, m)$ and $R(s, n, m)$ for $s = 2$ and $n = 3$ with a maximum on the symmetry axes.

The left and right building blocks of the unions are the same for T and R , because the symmetry axis of T bends around to the faces of the cube and the reflection in R is exactly the mirrored movement of the straight one in T . \square

4.4.3 Deformation graphs of cylinder knots

The difference functions $\Delta_i^\phi(\beta)$ can be displayed graphically to show whether the cylinder knot is stable or not. If we do this for all i simultaneously it is better to use $(\text{sign} [\Delta_i^\phi(2\pi)]) \cdot \Delta_i^\phi(\beta)$ because then all curves are positive for $\beta = 2\pi$ if ϕ is non-singular. Because of equation (4.3) the sign of $(\text{sign} [\Delta_i^\phi(2\pi)]) \cdot \Delta_i^\phi(\beta)$ is independent of ϕ .

We call a graph containing these normalized difference functions a *deformation graph*. All we have to do to check the stability of the knot is to observe if there is a neighbourhood $]0, \varepsilon]$ of zero in which all curves are positive or all are negative. Examples are:

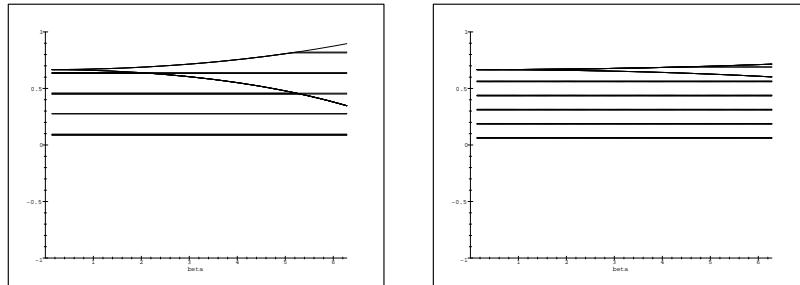


Figure 4.7: The deformation graphs of $Z(3, 11, 16)$ and $Z(3, 16, 11)$

Thus the knots $Z(3, 11, 16)$ and $Z(3, 16, 11)$ are strongly positive stable

and Corollary 4.4.3 yields $Z(3, 11, 16) = Z(3, 16, 11)$.

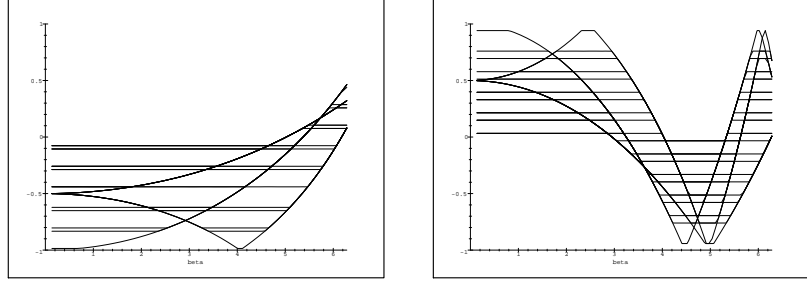


Figure 4.8: The deformation graphs of $Z(4, 11, 39)$ and $Z(4, 11, 119)$

The knot $Z(4, 11, 39)$ is negatively stable and the knot $Z(4, 11, 119)$ is positively stable. Corollary 4.4.5 gives the result $\det Z(4, 11, 39) = \det R(2, 11, 39)$.

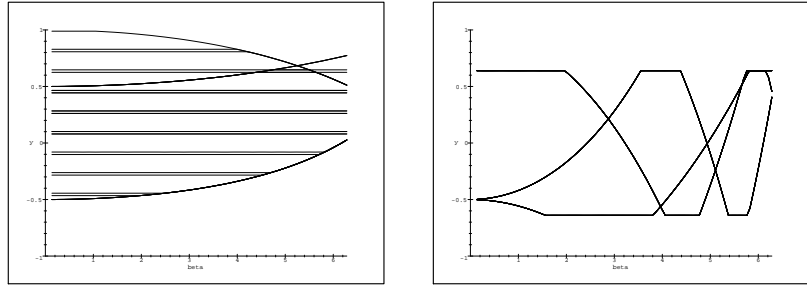


Figure 4.9: The deformation graphs of $Z(4, 11, 13)$ and $Z(4, 11, 121)$

From Figure 4.9 we get the information that $Z(4, 11, 13)$ and $Z(4, 11, 121)$ are not stable. The three lines in the second deformation graph result from the knot's periodicity. Therefore only three difference functions occur.

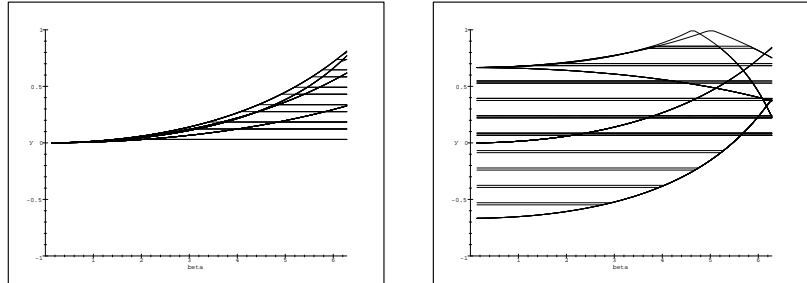


Figure 4.10: The deformation graphs of $Z(5, 13, 20)$ and $Z(6, 13, 14)$

In Figure 4.10 examples of deformations are shown for parameters s and m which are not coprime. In this case the knot $T(s, n, m)$ does not exist,

and there are difference functions converging to zero for $\beta \rightarrow 0$. The knot $Z(5, 13, 20)$ is strongly positive stable and the knot $Z(6, 13, 14)$ is not stable.

We give more information on the knots $Z(4, 11, m)$ for $m = 1, \dots, 156$. They are strongly positive stable for $m = 1, \dots, 12, 14, 15, 16, 18, 19, 20, 22, 23, 24, 28, 32, 36, 40, 44, 48$, positively stable (but not strongly positive stable) for $m = 119, 123, 127, 131$, negatively stable for $m = 39, 43, 47, 51, 55, 59, 60, 62, 63, 66, 67, 70, 71, 137, 141, 145, 149, 153$ and for the others not stable (they could be weakly stable but we did not check this.)

4.5 Additional parameters of deformation

In this section we mention two further possibilities to deform cylinder knots. For the first one we replace the cylinder by an annulus \times interval. The inner radius r is variable, see Figure 4.11.

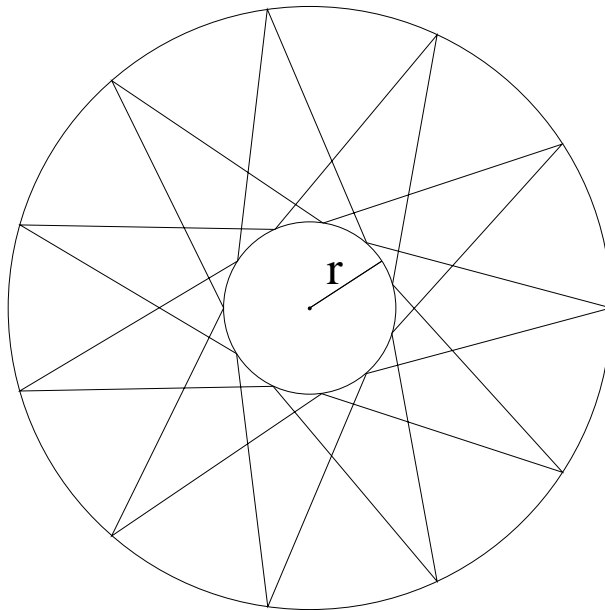


Figure 4.11: The projection of a billiard knot in a prism over an annulus with $s = 3$ and $n = 11$.

Another possibility is the study of prisms over ellipses. The excentricity ε and the starting point of the billiard curve are additional parameters. By the classical Theorem of Poncelet every point on the ellipse is starting point of a periodic billiard curve with s rotations and n reflections, if one point has this property. Furthermore all such periodic curves have the same length.

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